Superspace and Superfields

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We mainly follow section 4 of [1].

1 Superspace

We work in four dimension. $\mathcal{N}=1$ superspace is an extension of Minkowski space with four additional Grasmann coordinates $(x^\mu) \to (x^\mu, \theta_\alpha, \overline{\theta}_{\dot{\alpha}})$. Similarly to how Minkowski space can be viewed as a coset of the full Poincaré group modulu the Lorentz subgroup

$$\mathbb{R}^{1,3} = \frac{ISO(1,3)}{SO(1,3)}\,, \tag{1}$$

with the identification of coordinates with group elements

$$x^\mu \leftrightarrow e^{x^\mu P_\mu}\,, \tag{2}$$

superspace can be viewed as a coset of the superPoincaré group modulu the Lorentz subgroup

$$\mathcal{M}_{4|1} = \frac{\overline{Osp(4|1)}}{SO(1,3)}\,, \tag{3}$$

where the Lie Algebra of $Osp(4|\mathcal{N})$, for general $\mathcal{N}$, is a grade one graded Lie algebra $\mathfrak{osp}(4|\mathcal{N}) = \mathbb{L}_0 \oplus \mathbb{L}_1$ whose matrix representation can be written as

$$\begin{pmatrix} A_{4\times 4} & 0 \\ 0 & D_{\mathcal{N}\times\mathcal{N}} \end{pmatrix} + \begin{pmatrix} 0 & B_{4\times\mathcal{N}} \\ C_{\mathcal{N}\times 4} & 0 \end{pmatrix}\,, \tag{4}$$

with the first matrix corresponding to $\mathbb{L}_0$ and the second to $\mathbb{L}_1$ and $A \in \mathfrak{sp}(4)$, $D \in \mathfrak{o}(\mathcal{N})$. Thus, the first factor in the graded Lie algebra can be expressed as

$$\mathbb{L}_0 = \mathfrak{sp}(4) \otimes \mathfrak{o}(\mathcal{N})\,, \tag{5}$$

which inspires the name of the whole subalgebra. The bar signifies the Inonu-Wigner contraction that we do not explain here. More physically, the role of the matrices A, B, C, D is played by the standard superPoincaré generators as

$$A \to P_\mu\,, M_{\mu\nu}\,, \quad D \to Z^{IJ}\,, \quad B\,, C \to Q_I, \overline{Q}_I\,. \tag{6}$$

Going back the the special case $\mathcal{N}=1$, the identification of the superspace coordinates with the group elements is

$$(x^\mu, \theta_\alpha, \overline{\theta}_{\dot{\alpha}}) \leftrightarrow e^{x^\mu P_\mu} e^{\theta^\alpha Q_\alpha + \overline{\theta}_{\dot{\alpha}} \overline{Q}^{\dot{\alpha}}}\,. \tag{7}$$

Since θ and $\overline{\theta}$ are Grasmann, the Taylor expansion of the most general superfield (a function of the superspace coordinates) is most forth order in the θ's (2 θ's and 2 $\overline{\theta}$'s). If we know the superfiled at one point in superspace, we can generate the superfield at nearby points by

$$Y(x+\delta x, \theta+\delta\theta, \overline{\theta}+\delta\overline{\theta}) = e^{-i(\epsilon Q+\overline{\epsilon}\overline{Q})} Y(x,\theta,\overline{\theta}) e^{i(\epsilon Q+\overline{\epsilon}\overline{Q})}\,, \tag{8}$$

where we have adopted the convention to drop spinor indices and just assume $\epsilon Q = \epsilon^\alpha Q_\alpha$ and $\overline{\epsilon}_{\dot{\alpha}}\overline{Q}^{\dot{\alpha}}$. Using the Becker-Campbell-Hausdorff formula one can show that the variations of the coordinates are given in terms of the spinor parameters ϵ and $\overline{\epsilon}$ as

$$\begin{aligned} \delta x^\mu &= i\theta\sigma^\mu\overline{\epsilon} - i\epsilon\sigma^\mu\overline{\theta}\,, \\ \delta\theta^\alpha &= \epsilon^\alpha\,, \\ \delta\theta_{\dot{\alpha}} &= \overline{\epsilon}_{\dot{\alpha}}\,. \end{aligned} \tag{9}$$

2 Chiral Superfield

We define the covariant derivatives

$$\begin{aligned} D_\alpha &= \partial_\alpha + i\sigma^\mu_{\alpha\dot{\beta}}\overline{\theta}^{\dot{\beta}}\partial_\mu\,, \\ \overline{D}_{\dot{\alpha}} &= \overline{\partial}_{\dot{\alpha}} + i\theta^\beta\sigma^\mu_{\beta\dot{\alpha}}\partial_\mu\,. \end{aligned} \tag{10}$$

Using them we can define and chiral and an anti-chiral superfield as

$$\overline{D}_{\dot{\alpha}}\Phi = 0\,, \quad D_\alpha\Psi = 0\,. \tag{11}$$

Clearly, if Φ is chiral; $\overline{\Phi}$ is anti-chiral. Explicitly working out the chirality constraints (11) one can show that the most general form a chiral superfield is

$$\Phi = \phi + \sqrt{2}\theta\psi + i\theta\sigma^\mu\overline{\theta}\partial_\mu\phi - \theta\theta F - \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi\sigma^\mu\overline{\theta} - \frac{1}{4}\theta\theta\overline{\theta}\overline{\theta}\partial_\mu\partial^\mu\phi\,. \tag{12}$$

The fields $(\phi, \psi_\alpha; F)$ are functions of the standard Minkowski coordinates only and constitute precisely an $\mathcal{N}=1$ chiral multiplet (F is an auxilary field that can be integrated out). In terms of new coordinates

$$y^\mu = x^\mu + i\theta\sigma^\mu\overline{\theta}\,, \quad \overline{y}^\mu = x^\mu - i\theta\sigma^\mu\overline{\theta}\,, \tag{13}$$

the chiral superfield takes the simpler form

$$\Phi = \phi + \sqrt{2}\theta\psi - \theta\theta F\,. \tag{14}$$

3 Vector Superfield

A real, aka vector, superfield is one that obeys the condition

$$V = \overline{V}\,. \tag{15}$$

Explicitly working out this constraint on a generic superfield expanded up to four θ's we see that the most general vector superfield is given by

$$V = C + i\theta\chi - i\overline{\theta}\overline{\chi} + \theta\sigma^\mu\overline{\theta}v_\mu + \frac{i}{2}\theta\theta M - \frac{i}{2}\overline{\theta}\overline{\theta}M^* \\ + i\theta\theta\overline{\theta}\left(\overline{\lambda} + \frac{i}{2}\overline{\sigma}^\mu\partial_\mu\chi\right) - i\overline{\theta}\overline{\theta}\theta\left(\overline{\lambda} + \frac{i}{2}\sigma^\mu\partial_\mu\overline{\chi}\right) + \frac{1}{2}\theta\theta\overline{\theta}\overline{\theta}\left(D - \frac{1}{2}\partial_\mu\partial^\mu C\right). \tag{16}$$

One can immediately notice that $\Phi + \overline{\Phi}$ is a vector superfield and, further, that the transformation

$$V \to V + \Phi + \overline{\Phi} \tag{17}$$

is essentially a generalization of the gauge transformation for ordinary vector fields since on the vector component of V it acts as

$$v_\mu \to v_\mu - 2\partial_\mu \mathrm{Im}\phi\,. \tag{18}$$

Most of the degrees of freedom in (16) can be gauged away. A convenient gauge is the Wess-Zumino gauge under which the vector superfield takes the simple form

$$V_{\mathrm{WZ}} = \theta\sigma^\mu\overline{\theta}v_\mu + i\theta\theta\overline{\theta}\overline{\lambda} - i\overline{\theta}\overline{\theta}\theta\lambda + \frac{1}{2}\theta\theta\overline{\theta}\overline{\theta}D\,. \tag{19}$$

The fields $(\lambda_\alpha, v_\mu; D)$ constitute precisely the $\mathcal{N} = 1$ vector multiplet. Notice that $\overline{\lambda}$ is not an independent field since it can be obtained from λ and that D is again an auxilary filed. In the $y, \overline{y}$ coordinates introduced in (13) the vector superfield in the Wess-Zumino gauge takes the form

$$V_{\mathrm{WZ}} = \theta\sigma^\mu\overline{\theta}v_\mu + i\theta\theta\overline{\theta}\overline{\lambda} - i\overline{\theta}\overline{\theta}\theta\lambda + \frac{1}{2}\theta\theta\overline{\theta}\overline{\theta}(D - i\partial_\mu v^\mu)\,. \tag{20}$$

As evident from (17) so far we have only been considering $U(1)$ as a gauge group. To generalize to a general gauge group G, first promote the superfields (and the ordinary fields, of course) to Lie algebra elements

$$V = V_a T^a\,, \quad \Phi = \Phi_a T^a\,, \quad a = 1, \ldots, \mathrm{dim}G\,. \tag{21}$$

Then, note that the finite version of the gauge transformation (17) can be seen to be

$$e^V \to e^{-\overline{\Phi}} e^V e^{\Phi}\,. \tag{22}$$

Defining the covariant derivative

$$D_\mu = \partial_\mu - \frac{i}{2}[v_\mu, \cdot]\,, \tag{23}$$

replacing all ordinary derivatives in (16) with it and expanding the finite gauge transformation (22) to second order, we see that now the ordinary vector field transform as a non-abelian vector field should

$$v_\mu \to v_\mu - 2\partial_\mu \mathrm{Im}\phi + i[v_\mu, \mathrm{Im}\phi]\,. \tag{24}$$

4 Gaugino Superfield

In this section we directly work with non-abelian gauge group G. The vector superfield contains explicitly v^μ. This is not the most convenient for building actions. Preferably, there should be some superfield generalization of the filed strength $F_{\mu\nu}$. Consider the following chiral superfields

$$W_\alpha = -\frac{1}{4}\overline{DD}\, e^{-V} D_\alpha e^V \,, \quad \overline{W}_{\dot{\alpha}} = -\frac{1}{4} DDe^V \overline{D}_{\dot{\alpha}} e^{-V} \,. \tag{25}$$

They are called gaugino fields, for reasons to become clear momentarily. Under the gauge transformation (22) one can explicitly check that they transform as

$$W_\alpha \to e^{-\Phi} W_\alpha e^{\Phi} \,, \quad \overline{W}_{\dot{\alpha}} \to e^{-\overline{\Phi}} \overline{W}_{\dot{\alpha}} e^{\overline{\Phi}} \,. \tag{26}$$

Expanding the definitions (25) to second order in the exponent and working in the y-coordinates (13) one can write the gaugino field explicitly

$$W_\alpha = -i\lambda_\alpha + \theta_\alpha D + i(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu} + \theta\theta(\sigma^\mu D_\mu \overline{\lambda})_\alpha \,, \tag{27}$$

where the field strength

$$F_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu - \frac{i}{2}[v_\mu, v_\nu] \tag{28}$$

has appeared as desired. We see that the multiplet that composes the gaugino filed $(\lambda_\alpha, v_\mu; F)$ has lowest spin component the gaugino, hence the name. The gaugino filed is also called supersymmetric field strength because it is the carrier of $F_{\mu\nu}$.

References

[1] M. Bertolini, “Lectures on Supersymmetry.”
https://people.sissa.it/~bertmat/susycourse.pdf.