$SU(2)$ Seiberg-Witten Theory with One Flavour

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We closely follow sections 4 and 5 of [1]. In fact this note is nothing but a rewriting and reshuffling of aforementioned sections is a way that a coherent story emerges for the case of one hyper. All diagrams are heavily inspired, although not directly copied, from [1].

1 UV Lagrangian and Symmetry Breaking

Consider an $\mathcal{N} = 2$ with gauge group $SU(2)$, with a vector multiplet, parametrized by an $\mathcal{N} = 1$ vector superfield V and an $\mathcal{N} = 1$ chiral superfield Φ, in the adjoint and one hypermultiplet, paramatrized by two $\mathcal{N} = 1$ chiral multiplets Q^a and $\widetilde{Q}_a$, in the doublet and anti-doublet respectively, where $a = 1, 2$ is the $SU(2)$ index. This means that we have only one flavour, $N_f = 1$. The Lagrangian is given by

$$\begin{aligned}\mathcal{L} = &\left(\frac{\operatorname{Im}\tau}{4\pi}\int \mathrm{d}^4\theta\,\mathrm{tr}\,\Phi^\dagger e^{[V,\cdot]}\Phi + \int \mathrm{d}^2\theta\,\frac{-i}{8\pi}\tau\,\mathrm{tr}\,W_\alpha W^\alpha + cc.\right)\\ &+\int \mathrm{d}^4\theta\left(Q^\dagger e^V Q + \widetilde{Q}e^{-V}\widetilde{Q}^\dagger\right) + \left(\int \mathrm{d}^2\theta\,\widetilde{Q}\Phi Q + cc.\right) + \left(\int \mathrm{d}^2\theta\,\widetilde{Q}\mu Q + cc.\right),\end{aligned} \tag{1}$$

where the gaugino superfield W_α is essentially the field strength of V, μ is the bare mass parameter of Q and we have suppressed $SU(2)$ indices in the second line. Classically

$$\langle\Phi\rangle = \mathrm{diag}(a, -a)\,, \quad Q = \widetilde{Q} = 0 \tag{2}$$

gives a supersymmetric vaccum in the Coloumb branch. With $a \neq 0$ the gauge group is broken as

$$SU(2) \to U(1) \tag{3}$$

and the scalars of the hypermultiplet acquire mass

$$M_Q = |\pm a + \mu|\,. \tag{4}$$

For a general state the BPS mass formula reads

$$M \geq |na + ma_D + f\mu|\,, \tag{5}$$

where (n, m) are the magnetic and electric charges, f is the charge under the $U(1)$ flavour symmetry (Q has $f = 1$ and $\widetilde{Q}$ has $f = -1$), and a_D is the diagonal element of the dual vev.

2 Running of the Coupling Constant

We would need the running of the coupling constant for the case of 1 flavour and for pure SYM, as we shall immediately see. We begin with the case $N_f = 1$. The one-loop running is

$$\tau(a) = 2\tau_{UV} - \frac{6}{2\pi i}\log\frac{a}{\Lambda_{UV}} + \dots = -\frac{6}{2\pi i}\log\frac{a}{\Lambda_1} + \dots\,, \quad \Lambda_1^6 = \Lambda_{UV}^6 e^{4\pi i\tau_{UV}}\,. \tag{6}$$

This running has been calculated in the holomorphic scheme and is one-exact to all orders in perturbation theory. The ... stand for non-perturbative corrections in the form of instantons as we will soon see. Finally, we have defined Λ_1 via the UV scale Λ_{UV} and coupling τ_{UV}, but in fact it is is an invariant quantity to all orders in perturbation theory and can be readily evaluated at any other scale

$$\Lambda_1^6 = a^6 e^{2\pi i\tau(a)}\,. \tag{7}$$

It is called complexified dynamical scale and can be considered as a vev of some background chiral field. The index 1 is to remind us that we are dealing with one flavour. The humber 6 in the above equations appears from the combination

$$6 = 2(2C(\text{adj}) - 2N_f C(\text{fund})) = 2\left(2N - 2N_f\frac{1}{2}\right), \quad \text{with } N = 2\,, N_f = 1\,, \tag{8}$$

where the quantity $C(\rho)$ is defined via

$$\operatorname{tr}\rho(T^a)\rho(T^b) = C(\rho)\delta^{ab}\,, \tag{9}$$

where we normalize the generators as $\operatorname{tr} T^aT^b = \frac{1}{2}\delta^{ab}$. Then $C(\text{adj})$ equals the dual Coxeter number. For the case with no flavours the story is analogous, but the coefficient is not 6, but 8 due to the fact that $N_f = 0$. We obtain for the running

$$\tau(a) = 2\tau_{UV} - \frac{8}{2\pi i}\log\frac{a}{\Lambda_{UV}} + \dots = -\frac{8}{2\pi i}\log\frac{a}{\Lambda_0} + \dots\,, \quad \Lambda_0^4 = \Lambda_{UV}^4 e^{2\pi i\tau_{UV}}\,. \tag{10}$$

Now we focus on the running in its entirety. When $|\mu|$ is very large we expect a running of the coupling constant as in Figure 1. At first we have the full $SU(2)$ theory with one flavour. At scale $\sim |\mu|$ the chiral fields $Q, \widetilde{Q}$ decouple and we get pure $SU(2)$ theory. When the vev scale $|a|$ is reached the gauge group gets broken to $U(1)$. From the running we can determine a rough relation between the complexified dynamical scales of the theories with and without flavour. This is done by equating the runnings

$$\tau_0(E) = -\frac{6}{2\pi i}\log\frac{E}{\Lambda_1}\,, \quad \tau_1(E) = -\frac{8}{2\pi i}\log\frac{E}{\Lambda_0}\,, \tag{11}$$

where the indices 0 and 1 indicate the number of flavour, at the point $E = \mu$. We obtain

$$\Lambda_0^4 = \mu\Lambda_1^3\,. \tag{12}$$

3 The Dual Variable a_D and the Monodramy at Infinity

The dual variable a_D is given by the first derivative of the prepotential and τ is given by second derivative of the prepotential from where, as long as we keep $|a| \gg |\Lambda_1|$, we can calculate

$$a_D = \frac{\partial F}{\partial a}\,, \quad \tau = \frac{\partial^2 F}{\partial a^2} \implies a_D = -\frac{6a}{2\pi i}\log\frac{a}{\Lambda} + \dots\,. \tag{13}$$

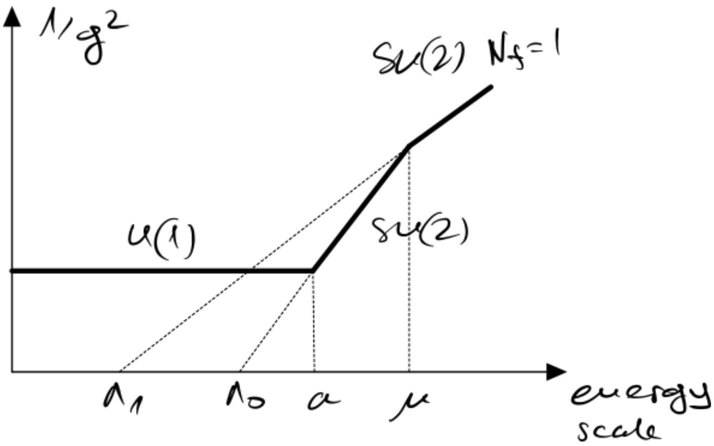

Figure 1: Running of the coupling constant

A gauge invariant way to label the supersymmetric vacua is to use the variable

$$u = \frac{1}{2} \left\langle \mathrm{tr}\, \Phi^2 \right\rangle = a^2 + \dots . \tag{14}$$

Adiabatically rotating around the entire complex plane

$$|u| \to e^{2\pi i} |u| \,, \tag{15}$$

we see that a and a_D transform as

$$\begin{pmatrix} a & a_D \end{pmatrix} \to \begin{pmatrix} a & a_D \end{pmatrix} \begin{pmatrix} -1 & 3 \\ 0 & -1 \end{pmatrix} . \tag{16}$$

From this we determine the monodramy at infinity

$$M_\infty = \begin{pmatrix} -1 & 3 \\ 0 & -1 \end{pmatrix} . \tag{17}$$

4 Singularities in the u-plane

4.1 Large and Intermediate μ

When μ is large and we consider scales smaller than μ we reduce to the case of pure $SU(2)$ SYM theory. This theory has two singularities in the u-plane

$$\text{pure } SU(2) \text{ singularities:} \qquad u \sim \pm \Lambda_0^2 \,. \tag{18}$$

This can be seen by considering the residual R-symmetry in the IR. The original R-symmetry in the IR is $U(1)_R$ and the fields in the vector multiplet have the R-charge assignments

$$\begin{array}{c|l} A_\mu & 0 \\ \lambda \quad \widetilde{\lambda} & 1 \,. \\ \Phi & 2 \end{array} \tag{19}$$

However, quantum mechanically, the rotation

$$\lambda \to e^{i\varphi}\lambda \tag{20}$$

is anomalous. Still, it can be compensated by a shift in the θ-angle

$$\theta_{UV} \to \theta_{UV} + 2(2C(\mathrm{adj}))\varphi = \theta_{UV} + 8\varphi\,, \tag{21}$$

where the numerical factor comes from the fact that λ and $\widetilde{\lambda}$ are in the adjoint. Thus $\varphi = \pi/4$ is a genuine symmetry which acts in the UV as

$$\theta_{UV} \to \theta_{UV} + 2\pi\,, \quad \Phi \to e^{i\pi/2}\Phi\,. \tag{22}$$

It is preserved in the IR and acts as

$$\theta_{IR} \to \theta_{IR} + 4\pi\,, \quad u = a^2 \to \left(e^{i\pi/2}a\right)^2 = -u\,. \tag{23}$$

Thus, singularities in the u plane come in pairs, and in fact one can show that there should be only two. Finally, in this regime the only scale is Λ_0, so the two singularities should be proportional to it as in (18).

If μ is intermediate, we expect a singularity at $\pm a \sim \mu$, since then the quanta of Q becomes massless, as can be seen from (4)

$$Q \text{ singularity for intermediate } \mu: \quad u \sim \mu^2\,. \tag{24}$$

Thus, in the large μ regime we have a total of three singularities, depicted in Figure

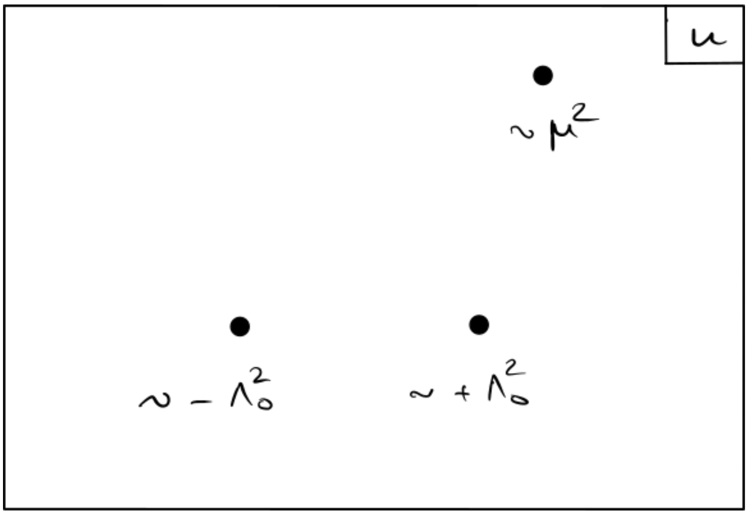

Figure 2: Singularities at large μ

4.2 Small μ

When $\mu = 0$ we can make use of the discrete leftover R-symmetry. The standard R-charge assignments of the full $U(1)_R$ IR R-symmetry, when we include one hyper are

$$\begin{array}{c|l} A_\mu & 0 \\ \lambda \quad \widetilde{\lambda} & 1 \\ \Phi & 2 \end{array} \,, \qquad \begin{array}{c|l} \psi_Q & -1 \\ Q \quad \widetilde{Q}^\dagger & 0 \\ \psi^\dagger_{\widetilde{Q}} & 1 \end{array} \,. \tag{25}$$

However the rotations

$$\lambda \to e^{i\varphi}\lambda \,, \psi_{Q,\widetilde{Q}} \to e^{-i\varphi}\psi_{Q,\widetilde{Q}} \tag{26}$$

are anomalous and must be compensated by a shift in the theta angle

$$\theta_{UV} \to \theta_{UV} + 2(2C(\text{adj}) - 2C(\text{fund}))\varphi = \theta_{UV} + 6\varphi \,. \tag{27}$$

Thus $\varphi = \pi/3$ is a genuine symmetry which acts in the UV as

$$\theta_{UV} \to \theta_{UV} + 2\pi \,, \Phi \to e^{2\pi i/3}\Phi \,. \tag{28}$$

It is preserved in the IR and acts as

$$\theta_{IR} \to \theta_{IR} + 4\pi \,, \quad u \to e^{4\pi i/3} u \,. \tag{29}$$

Thus, we conclude that the singularities for small μ should come in triplets

$$Q \text{ singularities for small } \mu : \quad u \sim \Lambda_1 \,, \omega\Lambda_1 \,, \omega^2\Lambda_1 \,, \tag{30}$$

and in fact one can show that there should be exactly 3 singularities.

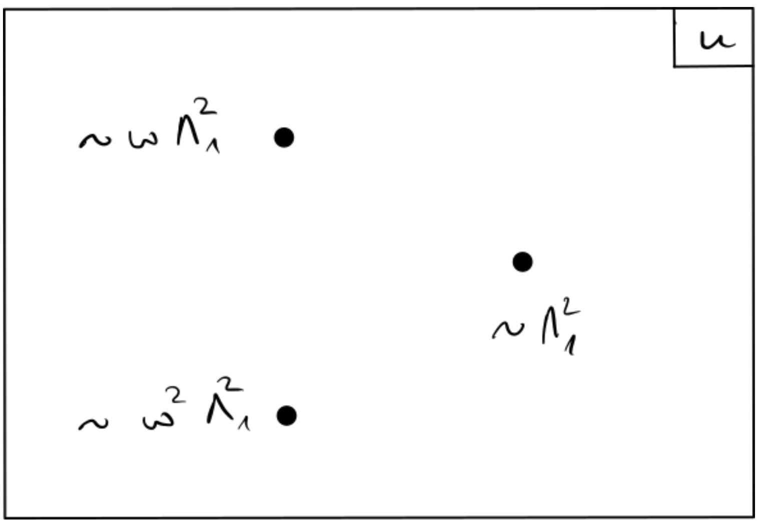

Figure 3: Singularities at small μ

5 Monodramies and the Theory Around the Singularities

5.1 Pure $SU(2)$

Consider for a moment the pure $SU(2)$ case with no flavours. In this case, we only have the singularities (18). It is clear that the monodramy at infinity is related to the two monodramies around these singularities as

$$M_\infty = M_+ M_- \,. \tag{31}$$

Further, from (23) we see that M_+ and M_- should be conjugate, by some $SL(2,\mathbb{Z})$ matrix, that we call X

$$M_- = X M_+ X^{-1} \,. \tag{32}$$

The monodramy problem is solved by

$$M_+ = STS^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \,, \quad M_- = T^2 STS^{-1} T^{-2} = \begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix} \,, \tag{33}$$

and the matrix $X = T^2$ corresponds nicely to a shift in θ_{IR} by 4π (23). Just by the monodramies, one can get a pretty good understanding of the physics of the singularities. Take the positive singularity $u = u_0 \sim +\Lambda_0$. We perform S-duality

$$a' = -a_D \,, a'_D = a \,, \tag{34}$$

exchanging the electric and magnetic charges. In the dual theory we can expand the vevs in a neighborhood around u_0

$$a' = c(u - u_0) \,, a'_D = \frac{a'}{2\pi i} \log c'(u - u_0) \,, \tag{35}$$

where we have used (13) and c, c' are some constants. For the coupling we find

$$\tau_D(a') = \frac{\partial a'_D}{\partial a'} \sim \frac{\log a'}{2\pi i} \,. \tag{36}$$

Comparing to (10) we find the behavior of of an effective $U(1)$ $\mathcal{N} = 2$ gauge theory coupled to one chiral multiplet with superpotential term

$$\int \mathrm{d}^2\theta \, \widetilde{Q} a' Q \,. \tag{37}$$

We see that as $u \to u_0$, τ_D blows up, which corresponds to $g_D \to 0$. The mass of the quantum of Q is given by

$$M_Q = |a'| = |a_D| \,. \tag{38}$$

Thus, we identify the charged chiral Q as the second quantized version of a monopole from the original theory. Indeed, heavy, non-perturbative monopoles in the original theory are light in the strongly coupled point. The behavior at $u = -u_0 \sim -\Lambda_0$ can be inferred by the discrete R-symmetry (23). We map the charges of the $u = u_0$ singularity with T^2 and obtain

$$T^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \,. \tag{39}$$

Thus at his singularity instead of a monopole, a dyon with charges $n = 2$ and $m = 1$ becomes light. For this reason the positive singularity is called a monopole point and the negative: a dyon point.

5.2 The Case of One Hyper

When we have the hyper present the monodramy equation reads

$$M_\infty = M_3 M_2 M_1 \,. \tag{40}$$

Moreover, in the small μ regime the monodramies are related by conjugation by an $SL(2,\mathbb{Z})$ matrix X

$$M_2 = X^{-1} M_1 X \,, \quad M_3 = X^{-2} M_1 X^2 \,. \tag{41}$$

A solution to this monodramy problem is given by

$$M_1 = STS^{-1} \,, \quad X = T \,. \tag{42}$$

We see that $M_1 \equiv M_+$, thus in the large μ regime the local physics around two of the singularities is exactly the same as in the pure $SU(2)$ case: a monopole and a dyon become light respectively. At the third point, one component of the doublet hyper $(Q, \widetilde{Q})$ becomes light. For all three singularities, the local description is that of a low energy $U(1)$ gauge theory coupled to one charged hyper.

In the small μ regime, the three singularities become related by the discrete R-symmetry (29) and it is no longer possible to distinguish which singularity come from the monopole, dyon or hyper points that we perfectly well distinguished in the large μ regime.

6 The Seiberg-Witten Curve

Introduce two auxilary complex variables (x, z) and consider the 1-dimensional complex curve in $\mathbb{C}^2$

$$\Sigma : \quad \frac{2\Lambda(x-\mu)}{z} + \Lambda^2 z = x^2 - u \,. \tag{43}$$

We compactify the point at infinity in the z-plane and call the resulting sphere C, such that we can view $x(z)$ as a coordinate on C and introduce the Sieberg-Witten differential

$$\lambda = x \frac{\mathrm{d}z}{z} \,. \tag{44}$$

The point $z = 0$ is a singularity but not a branch point and it has local behavior

$$\begin{aligned} x_+ &\sim \frac{2\Lambda}{z} - \mu + O(z) \,, \\ x_- &\sim +\mu + O(z) \,. \end{aligned} \tag{45}$$

Further

$$\mathrm{Res}(\lambda, z = 0) = \pm \mu \,. \tag{46}$$

There are a total of 4 branch points of the function $x(z)$ that we label $z = z_3, z_2, z_+, \infty$. We take the branch cuts to run between z_3 and z_2 and between z_+ and ∞. We see that the curve Σ is a two sheeted cover of C, depicted in Figure 4 Drawing the two cycles A and B as in Figure 4 we can

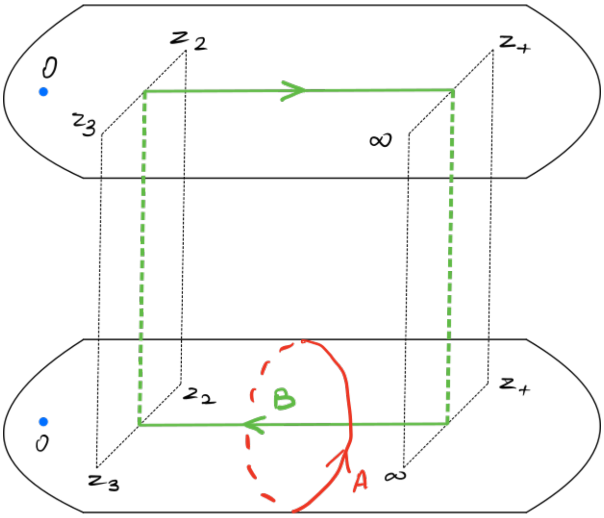

Figure 4: The two-sheeted sphere C with the branch points and the zero labelled.

declare that

$$a = \frac{1}{2\pi i}\int_A \lambda\,, \quad a_D = \frac{1}{2\pi i}\int_B \lambda\,. \tag{47}$$

To check that these declarations satisfy the physical requirements we have consider

$$\tau(a) = \frac{\partial a_D}{\partial a} = \frac{\partial a_D/\partial u}{\partial a/\partial u} = \frac{\int_A \partial\lambda/\partial u}{\int_B \partial\lambda/\partial u} = \frac{\int_A \omega}{\int_B \omega}\,, \tag{48}$$

where the differential

$$\omega \equiv \frac{\partial\lambda}{\partial u} = \frac{\mathrm{d}z}{xz} \tag{49}$$

is finite everywhere on Σ. Now consider a map from Σ to another complex plane, that is constructed by taking the endpoint P of a path starting at P_0 and integrating ω over it

$$t = \int_{P_0}^{P} \omega\,, \tag{50}$$

see Figure 5 . The above mapping is holomorphic, thus it preserves angles. As such the image of B is always to the left of the image of A. Since the complex parameter τ that lives in the t-plane is given by the ratio B/A, we see that

$$\operatorname{Im}\tau(a) > 0\,. \tag{51}$$

Thus, we see that (47) is indeed the physically correct assignment. Further, taking a closed path L on the torus that encircles A n times, B m times and the poles with residue $\pm\mu$ f times we obtain

$$\frac{1}{2\pi i}\int_L \lambda = na + ma_D + f\mu\,. \tag{52}$$

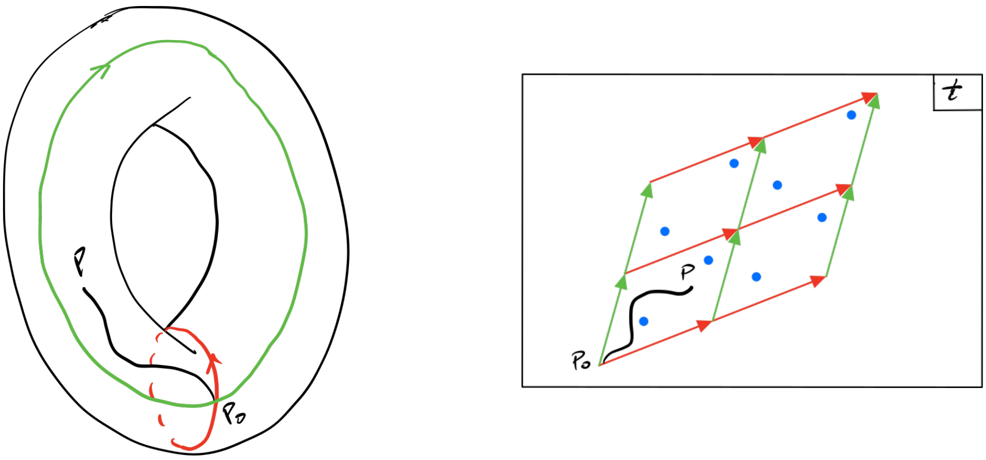

Figure 5: The path on the torus (the Seiberg-Witten curve), on the left, is mapped to the complex t-plane, on the right.

Let's check if we correctly reproduce the running of the coupling in the weakly coupled region. For simplicity, set $\mu = 0$, and assume $|u| \gg |\Lambda|$. Take the A cycle to have $|z| = 1$, and for the dominant contribution to the B-cycle, calculate the integral nearby the branch points $z_2 \sim \Lambda/\sqrt{u}$ and $z_+ \sim u/\Lambda^2$

$$\begin{aligned} a &= \frac{1}{2\pi i}\int_A \lambda \sim \sqrt{u}\,, \\ a_D &= \frac{1}{2\pi i}\int_B \lambda = \frac{2}{2\pi i}\int_{z_+}^{z_2} \lambda \sim -\frac{6}{2\pi i} a \log\frac{a}{\Lambda}\,. \end{aligned} \tag{53}$$

So we indeed have the correct behavior at small coupling. What is more, we can now perturbatively find all corrections from the curve which are of the form

$$\tau(a) = -\frac{6}{2\pi i}\log\frac{a}{\Lambda} + \sum_{k=0}^{\infty} c_k \left(\frac{\Lambda}{a}\right)^{6k}. \tag{54}$$

The second term above constitutes non-perturbative instanton corrections. Of course, it was known before the work of Seiberg and Witten that they take this general form, only now all of the coefficients c_k can be calculated explicitly from the curve, deriving the low energy effective theory in its entirety. Does the curve reproduce correctly the other two singularities? The branch points of the function $x(z)$ are determined when Σ has double roots, namely

$$z^3 + \frac{uz^2}{\Lambda^2} - \frac{2\mu z}{\Lambda} + 1 = 0\,. \tag{55}$$

Singularity in the u-plane is caused by two of the branch points above colliding, that is when the discriminant of the above equation vanishes

$$u^3 - \mu^2 u^2 + \Lambda^3 \mu u + \frac{27}{4}\Lambda^6 - 8\Lambda^3\mu^3 = 0\,. \tag{56}$$

When $\mu = 0$ this equation correctly reproduces the three singularities related by the discrete R-symmetry

$$u = c\Lambda^2\,, e^{2\pi i/3}c\Lambda^2\,, e^{4\pi i/3}c\Lambda^2\,. \tag{57}$$

When $|\mu| \gg \Lambda$ we find in for large and small u respectively

$$u \sim \mu^2\,, \quad u \sim \pm\sqrt{-8\Lambda^3\mu}\,. \tag{58}$$

We see that indeed the pure $SU(2)$ scale is given by

$$\Lambda_0^2 = \sqrt{\Lambda^3\mu}\,, \tag{59}$$

as we found earlier.

7 Notable Physical Effects

As μ moves along a semicircle with constant large $|\mu|$ the quark point rotates once, while the dyon and the monopole points rotate by $\frac{\pi}{2}$. If we now make $|\mu|$ small, the three singularities come closer in the small μ regime. Then we make it big to complete the semicircle. Studying the behavior of (56) carefully one can see that in this process exchanges the quark point and the monopole point

$$(Q, M, D) \to (M, D, Q)\,. \tag{60}$$

Calculate the discriminant of (56)

$$\mu^3 + \frac{27}{8}\Lambda^3 = 0\,. \tag{61}$$

Take $\mu = -\frac{3\Lambda}{2}$ as an explicit choice. The singularities in the u-plane collapse to two

$$u = -\frac{15\Lambda^2}{4}\,, \quad u = 3\Lambda^2\,. \tag{62}$$

In the curve three of the branch points colide as $z = -1$ and one remains at ∞. Thus $a = a_D = 0$, since the torus has degenerated. Using the BPS formula we see that simultaneously, magnetic and electic particles become light at this singularity.

References

[1] Y. Tachikawa, "N=2 supersymmetric dynamics for pedestrians," *arXiv:1312.2684 [hep-th]* **890** (2015) `1312.2684`. Comment: 190 pages. v2: many minor corrections thanks to the comments, and two new appendices. To be published in a book form.