Homogeneous Symmetric Spaces

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March 9, 2021

We mainly follow section 4.1 of [1].

Consider the unti sphere S^n embedded in $\mathbb{R}^{n+1}$. The isometry group is $G = SO(n+1)$ and the isotropy group is $H = SO(n)$, such that the sphere can be described as the coset

$$S^n = \frac{SO(n+1)}{SO(n)} \, . \tag{1}$$

A point on the sphere can be parametrized as

$$p(\alpha) = \exp \begin{pmatrix} 0 & \alpha^i \\ -\alpha_j & 0 \end{pmatrix} = \exp \begin{pmatrix} 0 & \alpha_1 & \cdots & \alpha_n \\ -\alpha_1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ -\alpha_n & 0 & \cdots & 0 \end{pmatrix} , \tag{2}$$

while a generic element of G is parametrized by

$$g(\alpha, \hat{h}) = p(\alpha) \exp\left(\hat{h}\right) = \exp \begin{pmatrix} 0 & \alpha^i \\ -\alpha_j & \hat{h} \end{pmatrix} , \quad \hat{h} \in \mathfrak{so}(n) \tag{3}$$

where $\alpha_i = \delta_{ij}\alpha^j$, $i, j = 1, \ldots, n$, and the Lie algebras $\mathfrak{so}(n+1)$ and $\mathfrak{so}(n)$ are given by $(n+1)\times(n+1)$ and $n \times n$ skew-symmetric matrices respectively. We see that indeed

$$\dim[SO(n+1)] = 1 + \cdots + n = \frac{(n+1)n}{2} \, . \tag{4}$$

The sphere is a symmetric and homogeneous space. The term symmetric will be explained in what follows and homogeneous refers to the fact that any point of the sphere can be reached by acting on that point with an element of G.

More generally, for the isometry group $G = SO(p, q)$ its elements satisfy

$$g^{-1} = \begin{pmatrix} \eta & 0 \\ 0 & 1 \end{pmatrix} g^T \begin{pmatrix} \eta & 0 \\ 0 & 1 \end{pmatrix} , \quad \eta = \mathrm{diag}(\underbrace{-1, \ldots, -1}_{p}, \underbrace{+1, \cdots + 1}_{q-1}) , \tag{5}$$

and the isotropy group is $H = SO(p, q-1)$, thus we see that the metric η is H-invariant. We can construct the following symmetric spaces

$$\begin{aligned} \mathbb{H}^n &= \frac{SO(n,1)}{SO(n)}\,, & \eta &= \mathrm{diag}(\underbrace{-1,\ldots,-1}_{n})\,, \\ \mathrm{dS}_n &= \frac{SO(1,n)}{SO(1,n-1)}\,, & \eta &= \mathrm{diag}(-1,\underbrace{+1,\ldots,+1}_{n-1})\,, \\ \mathrm{AdS}_n &= \frac{SO(n-1,2)}{SO(n-1,1)}\,, & \eta &= \mathrm{diag}(\underbrace{-1,\ldots,-1}_{n-1},+1)\,. \end{aligned} \tag{6}$$

The generators $\hat{h} \in \mathfrak{h}$, $\hat{g} \in \mathfrak{g}$ and $\hat{k} \in \overline{\mathfrak{h}}$, satisfy (schematically)

$$\begin{aligned} \left[\hat{h},\hat{h}\right] &= \hat{h} \implies & &\mathfrak{h} \text{ is a subalgebra } of \mathfrak{g}\,, \\ \left[\hat{h},\hat{k}\right] &= \hat{k} \implies & &\alpha_i \text{ rotate under } H\,, \\ \left[\hat{k},\hat{k}\right] &= \hat{h} \implies & &\text{coset space is symmetric}\,. \end{aligned} \tag{7}$$

In particular, for (5) to be satisfied if the coset representative $p(\alpha)$ is expressed by the first equality of (2), in the more general case of $G = SO(p,q)$, we have

$$\alpha_i = \eta_{ij}\alpha^j\,. \tag{8}$$

We can directly evaluate the exponent in of the first equality of (2) and obtain

$$p(\alpha) = \begin{pmatrix} \cos\alpha & \dfrac{\sin\alpha}{\alpha}\alpha^i \\ -\dfrac{\sin\alpha}{\alpha}\alpha_j & \delta^j_i + \alpha^i\alpha_j\dfrac{\cos\alpha - 1}{\alpha^2} \end{pmatrix}\,, \quad \alpha^2 = \eta_{ij}\alpha^i\alpha^j\,. \tag{9}$$

In the so called homogeneous coordinates of the embedding space

$$Y^A = \left(\pm\sqrt{1-y^2}, y^i = \frac{\sin(\alpha)}{\alpha}\alpha^i\right)\,, \tag{10}$$

the coset space embedding is described by the following relation

$$\left(Y^0\right)^2 + \eta_{ij}Y^iY^j = 1\,, \tag{11}$$

and a point on the coset is expressed as

$$p(y) = \begin{pmatrix} \pm\sqrt{1-y^2} & y^i \\ -y_j & \delta^i_j + y^iy_j\dfrac{\pm\sqrt{1-y^2}-1}{y^2} \end{pmatrix}\,, \tag{12}$$

where $\pm$ parametrize different parts of the coset space (for the sphere S^n, the north and south hemispheres respectively). Acting with a generic constant element $g \in G$ on a generic point $p(y)$, moves it somewhere along the sphere to a new location $p'(y)$. We can also move the point to that new location via a coordinate dependent isotropy transformation $h(y) \in H$ applied on some other point $p(y')$. That is

$$gp(y) = p(y')h(y)\,, \tag{13}$$

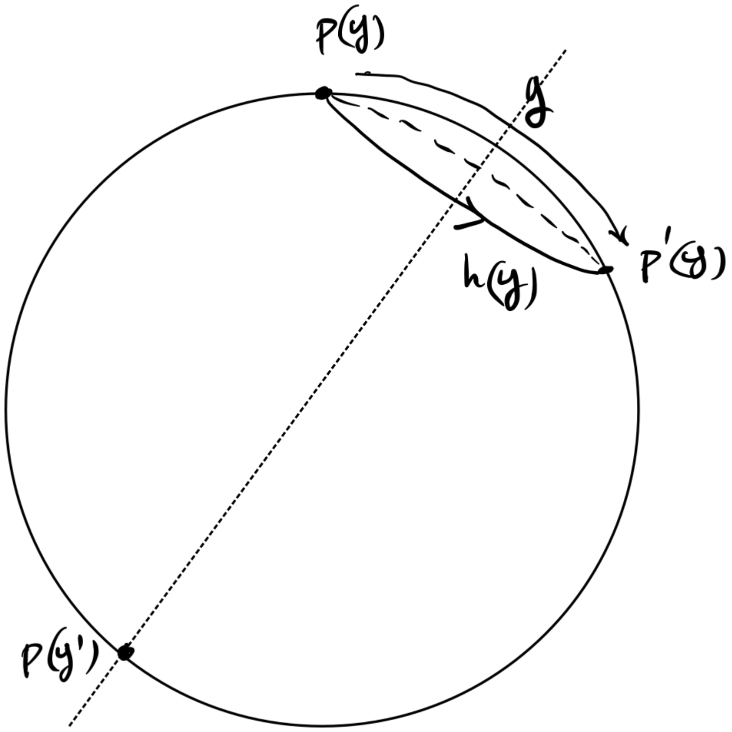

Figure 1: Constant isometry transformation of a point $p(y)$ by G, can equally well be executed via a coordinate dependent isotropy transformation centered at another point $p(y')$, as the diagram depicts for the special case of S^2.

where we have chose to apply the isometry transformation from the left and the isotropy transformation from the right. These transformations are schematically depicted for the sphere in Figure (1). The effect is a change of coordinates that satisfies the group multiplication laws. For an infinitesimal version of such change of coordinates $y^i \to y'^i = y^i + \xi^i(y)$, where $\xi^i(y)$ are the Killing vectors, we have

$$\xi^i(y)\partial_i p(y) = \hat{g}p(y) - p(y)\hat{h}(y)\,, \quad \hat{g} \in \mathfrak{g}\,,\ \hat{h} \in \mathfrak{h}\,. \tag{14}$$

To obtain the last equation we have expressed the group elements as

$$g \approx \mathbb{1} + \hat{g}\,, \quad h(y) \approx \mathbb{1} + \hat{h}(y)\,. \tag{15}$$

One might equally well use other other representations of the groups in question. For example, in the spinor representation of $SO(p, q+1)/SO(p, q)$ one has

$$p(\alpha) = \exp\left(\frac{1}{2}\Gamma_i\alpha^i\right) = \cos(\alpha/2)\mathbb{1} + i\frac{\sin(\alpha/2)}{\alpha}\alpha^i\Gamma_i\,, \tag{16}$$

where

$$\{\Gamma_i, \Gamma_j\} = -2\eta_{ij}\mathbb{1}\,. \tag{17}$$

In homogeneous coordinates a point on the coset takes the form

$$p(y) = \frac{1}{2}\Big(\sqrt{1+y} + \sqrt{1-y}\Big)\mathbb{1} + \frac{y^i \Gamma_i}{\sqrt{1+y} + \sqrt{1-y}} \,. \tag{18}$$

If we define a constant spinor on the "north pole" to be $\psi(0)$ the the following y dependent spinor

$$\psi(y) = p^{-1}\psi(y) \tag{19}$$

constitutes the Killing spinor on the coset space.

References

[1] B. de Wit, "Supergravity," *arXiv:hep-th/0212245* (Dec., 2002) `hep-th/0212245`. Comment: Lecture notes 2001 Les Houches Summerschool, LaTex, 147 pages. To be published in the proceedings.