Electromagnetic Duality

Vasil Dimitrov

February 21, 2021

We mainly follow section 1 of [1].

1 Electric charges, monopoles and dyons

Consider a $U(1)$ gauge field with gauge potential and field strength

$$A = A_\mu \, \mathrm{d}x^\mu \ , \quad F = \frac{1}{2} F_{\mu\nu} \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \ . \tag{1}$$

The gauge transformation is

$$A \to A + ig^{-1} \, \mathrm{d}g = A - \mathrm{d}\chi \ , \tag{2}$$

where $g = e^{i\chi} \in U(1)$ and $\chi \sim \chi + 2\pi$. A scalar filed gauge transforms as

$$\phi \to e^{in\chi} \phi \, , \tag{3}$$

and the covariant derivative is

$$D_\mu \phi = \partial_\mu \phi + inA_\mu \phi \, . \tag{4}$$

If we couple this Maxwell theory to an electric charge (with integer value n), that sits stationary at the origin, the action reads

$$S = \int \mathrm{d}^4 x \left(\frac{1}{2e^2} F_{\mu\nu} F^{\mu\nu} + A_\mu j_e^\mu \right) , \quad j_e^0 = n\delta^3(\vec{x}) \, , \, j_e^i = 0 \, . \tag{5}$$

From the field equation for A_ν we can read off

$$\frac{2}{e^2} \partial_\mu F^{\mu 0} = n\delta^3(\vec{x}) \implies \frac{4\pi}{e^2} \int_{S^2_\infty} \vec{E} \cdot \hat{n} = 2\pi n \, . \tag{6}$$

Defining the dual filed strength

$$\star F = \frac{1}{2} (\star F)_{\mu\nu} \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\mu \ , \quad (\star F)_{\mu\nu} = \widetilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \ , \tag{7}$$

we can rewrite the integral in (6) as

$$\frac{4\pi}{e^2} \int_{S^2_\infty} \vec{E} \cdot \hat{n} = \frac{4\pi}{e^2} \int_{S^2_\infty} \star F = 2\pi n \, . \tag{8}$$

Consider now a gauge field configuration, such that on the northern and southern semisphere gauge fields are related by gauge transformation on the the equator

$$A_N = A_S + ig^{-1}\,\mathrm{d}g\,. \tag{9}$$

We can evaluate the following integral

$$\int_{S^2_\infty} \vec{B}\cdot\hat{n} = \int_{S^2_\infty} F = \int_N F + \int_S F = \int_{\text{equator}} A_N - A_S = \int_0^{2\pi} \mathrm{d}\theta\,\frac{\mathrm{d}\chi}{\mathrm{d}\theta} = 2\pi m\,, \tag{10}$$

and conclude that this configuration constitutes a magnetic monopole with charge m, situated at the origin. A particle that carries both electric and magnetic charge is called a dyon. One can show that when we have two dyons, say with charges (n,m) and (n',m'), the overall z-component angular momentum is given by

$$L_z = \frac{\hbar}{2}(nm' - mn')\,. \tag{11}$$

Since, angular momentum is quantized, this also constitutes the Dirac quantization condition.

2 S-duality

Consider an “electric” theory coupled to electric current

$$S = \int \mathrm{d}^4x \left(\frac{1}{2e^2}F^{\mu\nu}F_{\mu\nu} - A_\mu j^\mu_e\right). \tag{12}$$

Viewing the field strength $F_{\mu\nu}$ as the fundamental field, the field equation and the Bianchi identity can be written in a gauge invariant way

$$\begin{aligned} \frac{2}{e^2}\partial_\mu F^{\mu\nu} &= j^\nu_e\,, & \frac{2}{e^2}\,\mathrm{d}F &= j_e\,,\\ \partial_\mu \widetilde{F}^{\mu\nu} &= 0\,, & \mathrm{d}\star F &= 0\,. \end{aligned} \tag{13}$$

Both of these equations can be obtained as field equations if we view $F^{\mu\nu}$ and $F^{\mu\nu}_D \equiv \alpha\widetilde{F}^{\mu\nu}$ as independent variables and introduce a Lagrange multiplier as

$$S = \int \mathrm{d}^4x \left(\frac{1}{2e^2}F^{\mu\nu}F_{\mu\nu} + A_\mu j^\mu_e + \frac{1}{e^2_D}A^D_\mu \partial_\mu F^{\mu\nu}_D\right). \tag{14}$$

Varying the action with respect to A_ν and A^D_ν we indeed obtain (13). The Lagrange multiplier term can be integrated by parts and the action takes the form

$$S = \int \mathrm{d}^4x \left(\frac{1}{2e^2}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2e^2_D}F^{\mu\nu}_D F^D_{\mu\nu} + A_\mu j^\mu_e\right). \tag{15}$$

At this point we can also introduce a magnetic current to the action

$$S = \int \mathrm{d}^4x \left(\frac{1}{2e^2}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2e^2_D}F^{\mu\nu}_D F^D_{\mu\nu} + A_\mu j^\mu_e + A^D_\mu j^\mu_m\right). \tag{16}$$

upon which the equations of motion take the form

$$\begin{aligned} \frac{2}{e^2}\partial_\mu F^{\mu\nu} &= j_e^\nu\,, & \frac{2}{e^2}\,\mathrm{d}F &= j_e\,, \\ \frac{2}{e_D^2}\,\partial_\mu F_D^{\mu\nu} &= -j_m^\nu\,, & \frac{2}{e_D^2}\,\mathrm{d}F_D &= -j_m\,. \end{aligned} \tag{17}$$

In the previous section we already saw how to derive charge quantization conditions from the field equation, namely:

$$\begin{aligned} \frac{2}{e^2}\,\mathrm{d}F = j_e \quad &\Longrightarrow \int_{S^2_\infty} \frac{4\pi}{e^2} \star F = 2\pi n\,, \\ \frac{2}{e_D^2}\,\mathrm{d}F_D = -j_m \quad &\Longrightarrow \int_{S^2_\infty} \frac{4\pi}{e_D^2} \star F_D = \int_{S^2_\infty} \frac{4\pi}{e_D^2}\alpha \star^2 F = -\int_{S^2_\infty} \frac{4\pi}{e_D^2}\alpha\, F = -2\pi m\,. \end{aligned} \tag{18}$$

Comparing to (10) we see that we must have $\frac{4\pi}{e_D^2}\alpha = 1$. Further the most sensible choice is $\alpha = \frac{4\pi}{e^2}$, such that

$$\int_{S^2_\infty} F_D = 2\pi n\,, \quad \int_{S^2_\infty} F = 2\pi m\,. \tag{19}$$

Thus, we have the relation

$$\frac{4\pi}{e^2}\frac{4\pi}{e_D^2} = 1\,. \tag{20}$$

We see that there is a duality of the equations of motion, called S-duality

$$\mathcal{S}: \quad \begin{pmatrix} F \\ F_D \end{pmatrix} \to \begin{pmatrix} F_D \\ F \end{pmatrix}, \quad \begin{pmatrix} e \\ e_D \end{pmatrix} \to \begin{pmatrix} e_D \\ e \end{pmatrix}, \quad \begin{pmatrix} j_e \\ j_m \end{pmatrix} \to \begin{pmatrix} -j_m \\ j_e \end{pmatrix}. \tag{21}$$

On the charges and on the complexified coupling

$$\tau = \frac{4\pi i}{e^2}\,, \quad \tau_D = \frac{4\pi i}{e_D^2} \tag{22}$$

S-duality acts as

$$\mathcal{S}: \quad \begin{pmatrix} n \\ m \end{pmatrix} \to S\begin{pmatrix} n \\ m \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} n \\ m \end{pmatrix} = \begin{pmatrix} -m \\ n \end{pmatrix}, \quad \tau \to S\tau = \tau_D = -\frac{1}{\tau}\,. \tag{23}$$

3 T-duality

Suppose that we have a neutral real scalar field ϕ, and that the gauge part of the action is given by

$$S = \int \mathrm{d}^4x \left(\frac{1}{2e(\phi)^2} F_{\mu\nu}F^{\mu\nu} + \frac{\theta(\phi)}{16\pi^2} F_{\mu\nu}\widetilde{F}^{\mu\nu} \right). \tag{24}$$

Maxwells equations now read

$$\begin{aligned} \mathrm{d}\left(\frac{4\pi}{e(\phi)^2} F + \frac{\theta(\phi)}{2\pi} \star F \right) &= 0\,, \\ \mathrm{d}\star F &= 0\,. \end{aligned} \tag{25}$$

As before we can calculate

$$\int_{S^2_\infty}\left(\frac{4\pi}{e(\phi)^2}\vec{E}+\frac{\theta(\phi)}{2\pi}\vec{B}\right)\cdot\hat{n}=2\pi n\,,\qquad \int_{S^2_\infty}\vec{B}\cdot\hat{n}=2\pi m\,. \tag{26}$$

We see that the same magnetic field as before satisfies Gauss' law, but there is modified conserved electric field

$$\vec{E}_{\text{conserved}}=\frac{4\pi}{e(\phi)^2}\vec{E}+\frac{\theta(\phi)}{2\pi}\vec{B}\,. \tag{27}$$

When we move around ϕ in the moduli space, the coupling and the θ-angle change. To keep the integer n constant, $\vec{E}$ acquires contribution from $\vec{B}$. This is called Witten effect. Under S-duality Maxwell's equations read

$$\begin{aligned}\mathrm{d}\left(\frac{4\pi}{e_D(\phi)^2}F_D+\frac{\theta_D(\phi)}{2\pi}\star F_D\right)&=0\,,\\ \mathrm{d}\star F_D&=0\,.\end{aligned} \tag{28}$$

In particular, we see that

$$\mathcal{S}:\quad F_D=\frac{4\pi}{e(\phi)^2}\star F-\frac{\theta(\phi)}{2\pi}F\,,\quad \tau_D(\phi)=-\frac{1}{\tau(\phi)}\,, \tag{29}$$

where we have extended the definition of the complexified coupling to incorporate the θ-angle

$$\tau(\phi)=\frac{4\pi i}{e(\phi)^2}+\frac{\theta(\phi)}{2\pi}\,,\quad \tau_D(\phi)=\frac{4\pi i}{e_D(\phi)^2}+\frac{\theta_D(\phi)}{2\pi}\,. \tag{30}$$

With the θ angle present there is another duality (on the quantum level) originating from the fact that under the shift $\theta(\phi)\to\theta(\phi)+2\pi$, the θ-term in the Euclidean path integral is shifted as

$$\exp\left(i\int \mathrm{d}^4x\frac{1}{8\pi}F_{\mu\nu}\widetilde{F}_{\mu\nu}\right)=\exp\left(i\pi\int \mathrm{d}^4x\,c_1(F)^2\right)=e^{i2\pi k}=1\,, \tag{31}$$

for integer k. We call this transformation T-duality. On the charges and complexified coupling it acts as

$$\mathcal{T}:\quad \begin{pmatrix}n\\ m\end{pmatrix}\to T\begin{pmatrix}n\\ m\end{pmatrix}=\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}n\\ m\end{pmatrix}=\begin{pmatrix}n+m\\ n\end{pmatrix}\,,\quad \tau\to S\tau=1+\frac{1}{\tau}\,, \tag{32}$$

where the action on the charges can be easily deduced from following the action on $\vec{E}_{\text{conserved}}$. We see that T-duality and S-duality, taken together generate an $SL(2,\mathbb{Z})$ action on the complexified coupling

$$\tau\to\frac{a\tau+b}{c\tau+d}\,,\quad \begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL(2,\mathbb{Z})\,. \tag{33}$$

References

[1] Y. Tachikawa, "N=2 supersymmetric dynamics for pedestrians," *arXiv:1312.2684 [hep-th]* **890** (2015) 1312.2684. Comment: 190 pages. v2: many minor corrections thanks to the comments, and two new appendices. To be published in a book form.