$\mathcal{N} = 2$ Actions

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We mainly follow section 2 of [1].

1 $\mathcal{N} = 2$ Vector Multiplet

The $\mathcal{N} = 2$ vector multiplet is composed of an $\mathcal{N} = 1$ chiral multiplet and an $\mathcal{N} = 1$ vector multiplet both in the adjoint of the gauge group G

$$\mathcal{N} = 2 \text{ vector multiplet} = \begin{cases} \mathcal{N} = 1 \text{ vector multiplet :} & \lambda_\alpha \,, \quad A_\mu \,, \\ \mathcal{N} = 1 \text{ chiral multiplet :} \quad \Phi \,, & \widetilde{\lambda}_\alpha \,. \end{cases} \tag{1}$$

The $\mathcal{N} = 1$ vector multiplet sits in the gaugino superfiled W_α and a vector superfield V (that we do not explicitly spell out)

$$W_\alpha = \lambda_\alpha + \frac{i}{2}\theta_\beta(\sigma^\mu)^\beta_{\dot\gamma}(\overline{\sigma}^\nu)^{\dot\gamma}_\alpha F_{\mu\nu} + D\theta_a + \dots \,, \tag{2}$$

and the $\mathcal{N} = 1$ chiral multiplet sits in the chiral superfield

$$\Phi = \Phi\Big|_{\theta=0} + 2\widetilde{\lambda}_\alpha\theta^\alpha + F\theta_\alpha\theta^\alpha + \dots \,, \tag{3}$$

where F and D are auxilary fields. Note that we denote with the same letter Φ both the chiral superfield and the ordinary scalar field Φ that enters in it. It should be clear from the context, which one we mean. Introducing the complexified coupling

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi} \tag{4}$$

the Lagrangian of the $\mathcal{N} = 2$ vector multiplet reads

$$\frac{\operatorname{Im}\tau}{4\pi}\int \mathrm{d}^4\theta \,\mathrm{tr}\, \Phi^\dagger e^{[V,\cdot]}\Phi + \int \mathrm{d}^2\theta\, \frac{-i}{8\pi}\tau\,\mathrm{tr}\, W_\alpha W^\alpha + cc. \,, \tag{5}$$

where the ratio between the two factors have been chosen such that the theory is invariant under the $SU(2)_R$ R-symmetry, rotating λ_α and $\widetilde{\lambda}_\alpha$ into each other. The supersymmetric vacua of this Lagrangian is obtained by demanding that the susy variations of the fermions vanish, which results into a constraint on the auxilary fields, and in turn, using their equations of motion. Since, we only have a kinetic term for Φ the nontrivial constraint is

$$\delta\lambda_\alpha = 0 \implies D_a = 0 \implies \frac{1}{g^2}\left[\Phi^\dagger, \Phi\right] = 0 \,, \tag{6}$$

where the index $a = 1, \dots, \mathrm{dim}G$ labels the generators of the Lie algebra.

2 $\mathcal{N}=2$ Hypermultiplet

The $\mathcal{N}=2$ hypermultiplet is composed of an $\mathcal{N}=1$ chiral and an $\mathcal{N}=1$ antichiral multiplet, both in a representation R of the gauge group

$$\mathcal{N}=2 \text{ chiral multiplet} = \begin{cases} \mathcal{N}=1 \text{ chiral multiplet :} & Q\,, \quad \psi_\alpha\,, \\ \mathcal{N}=1 \text{ antichiral multiplet :} & \widetilde{\psi}^\dagger_\alpha\,, \quad \widetilde{Q}^\dagger\,, \end{cases} \tag{7}$$

where the $SU(2)_R$ R-symmetry now rotates Q and $\widetilde{Q}^\dagger$ into each other. For definiteness consider $G=SU(N)$ and N_f hypers $Q^a_i, \widetilde{Q}^i_a$ in the fundamental N-dimensional representation, where $a=1,\ldots,N$ and $i=1,\ldots,N_f$. The gauge transformation then acts on the chiral superfields as

$$Q_i \to e^\Lambda Q_i\,, \quad \widetilde{Q}^i \to \widetilde{Q}^i e^{-\Lambda}\,, \tag{8}$$

where we are supressing the gauge indices and Λ is a traceless $N\times N$ matrix of chiral superfields. The Lagrangian of the $\mathcal{N}=2$ hypermultiplet in this case is

$$\int \mathrm{d}^4\theta \left(Q^{\dagger i} e^V Q_i + \widetilde{Q}^{\dagger i} e^{-V} \widetilde{Q}^\dagger_i\right) + \left(\int \mathrm{d}^2\theta\, \widetilde{Q}^i \Phi Q_i + cc.\right) + \left(\sum_i \int \mathrm{d}^2\theta\, \mu_i \widetilde{Q}^i Q_i + cc.\right), \tag{9}$$

where the presence of Φ and V reflects the fact the we have coupled the $\mathcal{N}=2$ hypermultiplet to the $\mathcal{N}=2$ vector multiplet, and the ratios of factors between the different terms and the fact that the mass matrix μ_i is diagonal both came from demanding invariance under $SU(2)_R$.

3 Vacua

The vacua of $\mathcal{N}=2$, that is the sum of the Lagrangians (5) and (9), is specified by

$$\begin{aligned} D_a = 0 &\quad\Longrightarrow\quad \frac{1}{g^2}[\Phi^\dagger,\Phi] + \left(Q_i Q^{\dagger i} - \widetilde{Q}^\dagger_i \widetilde{Q}^i\right)\Big|_{\mathrm{tr}} = 0\,, \\ F^{(\Phi)} = 0 &\quad\Longrightarrow\quad Q_i \widetilde{Q}^i\Big|_{\mathrm{tr}} = 0\,, \\ F^{(Q_i)} = 0 &\quad\Longrightarrow\quad \Phi Q_i + \mu^j_i Q_j = 0\,, \\ F^{(\widetilde{Q}^i)} = 0 &\quad\Longrightarrow\quad \Phi \widetilde{Q}^i + \mu^i_j \widetilde{Q}^j = 0\,, \end{aligned} \tag{10}$$

where for an $N\times N$ matrix X we have defined

$$X\Big|_{\mathrm{tr}} = X - \frac{1}{N}\,\mathrm{tr}\,X\,. \tag{11}$$

The full scalar potential is a weighted sum of the absolute values squared of the left hand sides of (10). There are two important subspaces of the vacuum moduli

$$\text{Coloumb branch :} \qquad [\Phi^\dagger,\Phi] = 0\,, \qquad Q = \widetilde{Q} = 0\,, \tag{12}$$

$$\text{Higgs branch :} \qquad \left(Q_i Q^{\dagger i} - \widetilde{Q}^\dagger_i \widetilde{Q}^i\right)\Big|_{\mathrm{tr}} = 0\,, \quad Q_i\widetilde{Q}^i\Big|_{\mathrm{tr}} = 0\,, \qquad \Phi = 0\,, \quad \text{for } \mu^j_i = 0\,. \tag{13}$$

For concreteness, we explore here the Coloumb branch of an $SU(2)$ gauge theory. From (12) we see that we can diagonalize the scalar part of the chiral superfield Φ

$$\langle \Phi \rangle = \mathrm{diag}(a, -a)\,. \tag{14}$$

When $a \neq 0$, we have a non-zero vev of a the chiral superfield and the following breaking of the gauge group occurs

$$SU(2) \to U(1)\,. \tag{15}$$

The first term in (5) contains a covariant derivative (modulus squared), therefore a term like

$$\frac{1}{g^2}\,\mathrm{tr}\left[A_\mu, \langle \Phi \rangle\right]^2\,, \tag{16}$$

which gives a mass to the vector field. Explicitly, the $SU(2)$ vector field

$$A_\mu^{SU(2)} = \begin{pmatrix} A_\mu^{U(1)} & W_\mu^+ \\ W_\mu^- & -A_\mu^{U(1)} \end{pmatrix} \tag{17}$$

is decomposed of $U(1)$ parts that remain massless, and W-bosons that acquire masses that can be read out from

$$\left[\begin{pmatrix} 0 & W_\mu^+ \\ 0 & 0 \end{pmatrix}, \langle \Phi \rangle\right] = -2a \begin{pmatrix} 0 & W_\mu^+ \\ 0 & 0 \end{pmatrix}\,, \quad \left[\begin{pmatrix} 0 & W_\mu^+ \\ 0 & 0 \end{pmatrix}, \langle \Phi \rangle\right] = -2a \begin{pmatrix} 0 & W_\mu^+ \\ 0 & 0 \end{pmatrix} \tag{18}$$

to be

$$M_{W^\pm} = |2a|\,. \tag{19}$$

The hypermultiplet scalars $\widetilde{Q}^i, Q_i$ also acquire mass via

$$\widetilde{Q}^i \langle \Phi \rangle Q_i + \mu_i \widetilde{Q}^i Q_i\,, \tag{20}$$

which can be readily read out to be

$$M_{Q_i} = |\pm a + \mu_i|\,. \tag{21}$$

for the two components in the $SU(2)$ fundamental representation.

4 BPS Bound

In the $\mathcal{N} = 2$ supersymmetry algebra the central charge Z appears as

$$\left\{Q_\alpha^I, Q_\beta^J\right\} = \epsilon^{IJ} \epsilon_{\alpha\beta} Z\,, \tag{22}$$

where $I, J = 1, 2$ are the $\mathcal{N} = 2$ indices. One can show that a mass of a state in the spectrum obeys

$$M \geq |Z|\,. \tag{23}$$

This is called the BPS bound and states obeying it are called BPS states. BPS states live in short multiplets that are generically protected even at the quantum level. In a $U(1)$ theory Z can be generically expressed as a linear combination of the electric (n), magnetic (m) and flavour charges (f_i), thus

$$M \geq \left| na + ma_D + \sum_i \mu_i f_i \right| . \tag{24}$$

In the weakly coupled regime the coefficients can be identified from calculating the masses of broken gauge bosons, magnetic monopoles and by looking at the mass term of hypers in the Lagrangian

$$a = \langle \Phi \rangle_{11} \ , \quad a_D = 2\tau a \, , \quad \mu_i = \text{mass term of } Q_i \, , \tag{25}$$

where τ is the complexified coupling. In the strong coupling regime, there is no sense in which the coefficient a can be thought of the diagonal entry of the vev of some filed Φ. Rather, (24) should be considered the definition of the coefficients (a, a_D, μ_i).

5 Low Energy Effective Lagrangian

Consider a breaking of the gauge group $G \to U(1)^n$. The n $\mathcal{N}=2$ vector multiplets are given be

$$\begin{cases} \mathcal{N}=1 \text{ vector multiplets}: & \lambda_{\alpha i} \, , \quad A_{\mu i} \, , \\ \mathcal{N}=1 \text{ chiral multiplets}: & a_i \, , \quad \widetilde{\lambda}_{\alpha i} \, , \end{cases} \tag{26}$$

where $i = 1, \ldots, n$. The $\mathcal{N}=1$ supersymmetric Lagrangian is given by

$$\frac{1}{8\pi} \int \mathrm{d}^4\theta \, K(\overline{a}, a) + \int \mathrm{d}^2\theta \, \frac{-i}{8\pi} \, \tau^{ij}(a) W_{\alpha i} W_j^{\alpha} + cc. \, , \tag{27}$$

where we have allowed for a non-trivial Kähler potential and the complexified gauge coupling has turned into a matrix with non-trivial (but holomorphic) dependence on a_i. For this Lagrangian to be a valid $\mathcal{N}=2$ Lagrangian it should respect the $SU(2)_R$, rotating $\lambda_{\alpha i}$ and $\widetilde{\lambda}_{\alpha i}$. We enforce this by equating the respective kinetic terms in (27)

$$\frac{\tau^{ij}(a) - \overline{\tau}^{ij}(\overline{a})}{4\pi i} = \frac{1}{4\pi} \frac{\partial^2 K(\overline{a}, a)}{\partial a_i \partial \overline{a}_j} \, . \tag{28}$$

At least locally, we can define a holomorphic function $F(a)$ (and its respective antiholomorphic one $\overline{F}(\overline{a})$), in terms of which we can express both τ^{ij} and K, such that (28) is satisfied

$$\tau^{ij} = \frac{\partial F}{\partial a_i a_j} \, , \quad \overline{\tau}^{ij} = \frac{\partial \overline{F}}{\partial \overline{a}_i \overline{a}_j} \, , \quad K = i\left(\frac{\partial \overline{F}}{\partial \overline{a}_i} a_i - \overline{a}_i \frac{\partial F}{\partial a_i} \right) . \tag{29}$$

The holomorphic function $F(a)$ is called prepotential and a Kähler potential that can be expressed in such manner in terms of a prepotential is called special Kähler. At weak coupling the variable a_i can be thought of as a diagonal entry in the vev of some chiral field Φ in the UV vector multiplet, as indeed consider a single hypermultiplet $Q, \widetilde{Q}$ charged only under the i-th vector multiplet in the IR theory, from the superpotential term

$$\widetilde{Q} a_i Q \implies M_Q = |a_i| \, . \tag{30}$$

While the combination

$$\frac{\partial F}{\partial a_i} \equiv a_D^i \tag{31}$$

can be thought as the vev diagonal element of the S-dual theory. This can be seen by considering the bosonic Lagrangian

$$\frac{\mathrm{Im}\,\tau^{ij}}{4\pi}\partial_\mu \overline{a}_i \partial^\mu a_j + \frac{\mathrm{Im}\,\tau^{ij}}{8\pi} F_{\mu\nu i} F^{\mu\nu j} + \frac{\mathrm{Re}\,\tau^{ij}}{8\pi} F_{\mu\nu i} \widetilde{F}^{\mu\nu j} \,, \tag{32}$$

and noting that under S-duality the gauge part goes to

$$\frac{\mathrm{Im}\,\tau_{Dij}}{8\pi} F^i_{D\mu\nu} F_D^{\mu\nu j} + \frac{\mathrm{Re}\,\tau_{Dij}}{8\pi} F^i_{D\mu\nu} \widetilde{F}_D^{\mu\nu j} \,, \tag{33}$$

where

$$\tau_{Dij} = -\frac{1}{\tau^{ij}} \,. \tag{34}$$

This means that the scalar part of (33) should equal the scalar part of the dualized Lagrangian

$$\mathrm{Im}\,\tau^{ij} \partial_\mu \overline{a}_i \partial^\mu a_j = \mathrm{Im}\,\tau_{Dij} \partial_\mu \overline{a}^i_D \partial^\mu a^j_D \,. \tag{35}$$

The last equation can be rewritten in terms of the prepotential

$$\partial_\mu \frac{\partial F}{\partial a_i} \partial^\mu \overline{a}_i - \partial_\mu \frac{\partial \overline{F}}{\partial \overline{a}_i} \partial^\mu a_i = \partial_\mu \overline{a}^i_D \partial^\mu \frac{\partial F}{\partial a^i_D} - \partial_\mu a^i_D \partial^\mu \frac{\partial \overline{F}}{\partial \overline{a}^i_D} \,, \tag{36}$$

and since F is holomorphic, the terms above should match independently, from where we indeed conclude that

$$a^i_D = \frac{\partial F}{\partial a_i} \,, \quad a_i = \frac{\partial F}{\partial a^i_D} \,. \tag{37}$$

Thus, we have n $\mathcal{N} = 2$ dual vector multiplets

$$\begin{cases} \mathcal{N} = 1 \text{ dual vector multiplets :} & \lambda^i_{D\alpha} \,, \quad A^i_{D\mu} \,, \\ \mathcal{N} = 1 \text{ dual chiral multiplets :} \quad a^i_D \,, & \widetilde{\lambda}^i_{D\alpha} \,. \end{cases} \tag{38}$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}x\, \hat{f}(k) e^{ikx} \tag{39}$$

References

[1] Y. Tachikawa, "N=2 supersymmetric dynamics for pedestrians," *arXiv:1312.2684 [hep-th]* **890** (2015) 1312.2684. Comment: 190 pages. v2: many minor corrections thanks to the comments, and two new appendices. To be published in a book form.