$\mathcal{N} = 1$ Actions

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We mainly follow section 5 of [1].

Supersymmetric actions can be constructed as integrals over superspace of functions of superfields. Integrals over the Grasmann variables θ and $\overline{\theta}$ are related to derivatives as

$$\int \mathrm{d}^2\theta = \frac{1}{4}\epsilon^{\alpha\beta}\partial_\alpha\partial_\beta\,, \quad \int \mathrm{d}^2\overline{\theta} = -\frac{1}{4}\epsilon^{\dot{\alpha}\dot{\beta}}\partial_{\dot{\alpha}}\partial_{\dot{\beta}}\,. \tag{1}$$

We denote the following integral over θ's and $\overline{\theta}$'s as

$$\int \mathrm{d}^2\theta\,\mathrm{d}^2\overline{\theta} = \int \mathrm{d}^4\theta\;. \tag{2}$$

1 $\mathcal{N} = 1$ Actions

1.1 Matter Actions

The most general $\mathcal{N} = 1$ matter Lagrangian involving four Grasmann integrations and one chiral superfield is

$$\int \mathrm{d}^4\theta\, K(\Phi, \overline{\Phi})\,, \tag{3}$$

where the function $K(\Phi, \overline{\Phi})$ is called Kähler potential. It can be shown that if we want a supersymmetric invariant theory, with equations of motion of no higher than second order, the Kähler potential is constrained to

$$K(\Phi, \overline{\Phi}) = \sum_{n,m=1}^{\infty} c_{mn}\overline{\Phi}^m\Phi^n\,, \quad c_{mn} = c^*_{nm}\,, \quad c_{mn} = 0\,,\, n \neq m\,. \tag{4}$$

Such a term in the Lagrangian is called a D-term. Note that there are no terms in the sum above with $n = 0$ or $m = 0$, since the following transformation is a symmetry of the theory

$$K(\Phi, \overline{\Phi}) \to K(\Phi, \overline{\Phi}) + \Lambda(\Phi) + \overline{\Lambda}(\overline{\Phi})\,, \tag{5}$$

meaning that such contribution can be "gauged" away. The final condition in (4) comes from demanding that $K(\Phi, \overline{\Phi})$ has R-charge 0. Remember that

$$R[\theta] = 1\,, \quad R[\overline{\theta}] = -1\,, \quad R[\mathrm{d}\theta] = -1\,, \quad R[\mathrm{d}\overline{\theta}] = 1\,, \tag{6}$$

thus for the whole action to have R charge 0, the Kähler potential shall obey

$$R[K] = 0\,. \tag{7}$$

If we want a renormalizable theory, all c_{mn}'s should vanish, except c_{11}. The resulting renormalizable theory is a free theory.

The most general $\mathcal{N} = 1$ matter Lagrangian involving two Grasmann integrations and one chiral superfield is

$$\int \mathrm{d}^2\theta\, W(\Phi) + \int \mathrm{d}^2\overline{\theta}\, \overline{W}(\overline{\Phi})\,, \tag{8}$$

where the function $W(\Phi)$ is holomorphic in Φ. It is called superpotential and generically it is constrained to be of the form

$$W(\Phi) = \sum_{n=1}^{\infty} a_n \Phi^n\,. \tag{9}$$

Further, for the whole action to have R-charge 0, then the superpotential shall have

$$R[W] = 2\,. \tag{10}$$

If we want a renormalizable theory the upmost term in (9) shall be Φ^3. A term like (8) in the Lagrangian is called an F-term.

All that was said so far can be generalized to a set of n chiral fields $\Phi^{i=1,\dots,n}$. The renormalizable matter Lagrangian is then given by

$$\begin{aligned} \mathcal{L}^{\text{renorm.}}_{\text{matter}} &= \int \mathrm{d}^4\theta\, K(\Phi^i, \overline{\Phi}_i) + \int \mathrm{d}^2\theta\, W(\Phi^i) + \int \mathrm{d}^2\overline{\theta}\, \overline{W}(\overline{\Phi}_i)\,, \\ K(\Phi^i, \overline{\Phi}_i) &= \overline{\Phi}_i \Phi^i\,, \quad W(\Phi^i) = a_i \Phi^i + \frac{1}{2} m_{ij} \Phi^i \Phi^j + \frac{1}{3} g_{ijk} \Phi^i \Phi^j \Phi^k\,. \end{aligned} \tag{11}$$

This Lagrangian can be readily evaluated in terms of the basic fields

$$\mathcal{L}^{\text{renorm.}}_{\text{matter}} = \partial_\mu \overline{\phi}_i \partial^\mu \phi^i + \frac{i}{2}\left(\partial_\mu \psi^i \sigma^\mu \overline{\psi}_i - \psi^i \sigma^\mu \partial_\mu \overline{\psi}_i\right) + \overline{F}_i F^i - W_i F^i - \frac{1}{2} W_{ij} \psi^i \psi^j - W^i \overline{F}_i - \frac{1}{2} W^{ij} \overline{\psi}_i \overline{\psi}_j\,, \tag{12}$$

where we have defined

$$W_i = \frac{\partial W(\phi)}{\partial \phi^i}\,, \quad W^i = \overline{W}_i\,, \quad W_{ij} = \frac{\partial^2 W(\phi)}{\partial \phi^i \partial \phi^j}\,, \quad W^{ij} = \overline{W}_{ij}\,. \tag{13}$$

Notice that when the expressions above are evaluated the argument of W is taken to be ϕ instead of Φ. We can eliminate the auxilary field using its equations of motion

$$\overline{F}_i = W_i\,, \quad F^i = W^i\,, \tag{14}$$

and arrive at the on-shell Lagrangian

$$\mathcal{L}^{\text{renorm.}}_{\text{matter, on-shell}} = \partial_\mu \overline{\phi}_i \partial^\mu \phi^i + \frac{i}{2}\left(\partial_\mu \psi^i \sigma^\mu \overline{\psi}_i - \psi^i \sigma^\mu \partial_\mu \overline{\psi}_i\right) - W_i W^i - \frac{1}{2} W_{ij} \psi^i \psi^j - \frac{1}{2} W^{ij} \overline{\psi}_i \overline{\psi}_j\,, \tag{15}$$

from where we can read off the scalar potential

$$V(\phi^i, \overline{\phi}_i) = W_i W^i = \sum_{i=1}^{n} \left| \frac{\partial W}{\partial \phi^i} \right|^2 = \overline{F}_i F^i \,, \tag{16}$$

where the final expression shall be taken off-shell.

The whole story be generalized for a σ-model, aka non-renormalizable Lagrangian, which might be relevant for constructing low energy effective theories. We just give the final result which reads for the on-shell action

$$\mathcal{L}^{\sigma\text{-model}}_{\text{matter, on-shell}} = K^i_j \partial_\mu \phi^i \partial^\mu \overline{\phi}_j - (K^{-1})^i_j W_i W^j + \text{fermions} \,, \tag{17}$$

where similarly to the superpotential we have defined

$$K^i_j = \frac{\partial^2 K(\phi, \overline{\phi})}{\partial \phi^i \partial \overline{\phi}_j} \,. \tag{18}$$

This quantity is Hermitian $K^j_i = K^{i\,\dagger}_j$, since $K(\phi, \overline{\phi})$ is a real function. Moreover it is positive definite and non-singular. Thus, it can be interpreted as a metric on the scalar manifold $\mathcal{M}$. It is known as Kähler metric. The scalar potential is given by

$$V(\phi, \overline{\phi}) = (K^{-1})^i_j W_i W^j \,. \tag{19}$$

1.2 Gauge Action

To construct the most general $\mathcal{N} = 1$ action involving only gauge fields one uses the gaugino superfiled, since it carries the field strength. The superYang-Mills action involving one gaugino field is given by

$$\begin{aligned} \mathcal{L}_{\text{SYM}} &= \frac{1}{32\pi} \text{Im}\left(\tau \int \mathrm{d}^2\theta \, \text{Tr} W^\alpha W_\alpha \right) \\ &= \text{Tr}\left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - i\lambda \sigma^\mu D_\mu \overline{\lambda} + \frac{1}{2} D^2 + \frac{\theta_{YM}}{32\pi^2} \, g^2 F_{\mu\nu} \widetilde{F}^{\mu\nu} \right] , \end{aligned} \tag{20}$$

where the trace is over the matrices that enter in the Lie algebra of the gauge group in the adjoint representation and we have introduced the complexified coupling

$$\tau = \frac{4\pi i}{g^2} + \frac{\theta_{YM}}{2\pi} \,. \tag{21}$$

This Lagrangian might look like an F-term but in fact, it is a D-term since we can write

$$\int \mathrm{d}^2\theta \, W^\alpha W_\alpha = \int \mathrm{d}^4\theta \, D^\alpha V \, W_\alpha \,. \tag{22}$$

1.3 Gauge Field Coupled to Matter

Suppose that the matter fields transform in a representation R of a semi-simple gauge group G, such that the Lie algebra generators are expressed as matrices $(T^a_R)^i_j$, with $i, j = 1, \ldots, \dim R$. A chiral superfield gauge transforms as

$$\Phi \to e^{i\Lambda_R} \Phi \,, \quad \Lambda_R = \Lambda_a T^a_R \,. \tag{23}$$

The most general $\mathcal{N} = 1$ gauge invariant and susy invariant matter action coupled to a vector multiplet contains the following D- and F- terms

$$\mathcal{L}_{\text{matter}} = \int \mathrm{d}^4\theta\, \overline{\Phi} e^{2gV_R} \Phi + \int \mathrm{d}^2\theta\, W(\Phi) + \int \mathrm{d}^2\overline{\theta}\, \overline{W}(\overline{\Phi})\,, \tag{24}$$

where we are suppressing the gauge group indices, V_R is a vector superfield in the representation R, and in the superpotential a term like

$$c_{i_1,\dots i_n} \Phi^{i_1} \dots \Phi^{i_n} \tag{25}$$

is only allowed if $c_{i_1,\dots i_n}$ is an invariant tensor of the gauge group and $R^{\times n}$ contains the singlet representation of G. The total Lagrangian $\mathcal{L} = \mathcal{L}_{\text{SYM}} + \mathcal{L}_{\text{matter}}$ can be explicitly evaluated exactly as we did before and the scalar potential can be read out

$$V(\phi, \overline{\phi}) = W_i W^i + \frac{g^2}{2} \sum_a \left| \overline{\phi}_i (T^a)^i_j \phi^j \right|^2 . \tag{26}$$

References

[1] M. Bertolini, “Lectures on Supersymmetry.”
`https://people.sissa.it/~bertmat/susycourse.pdf`.